

SUSTAINED TURBULENCE IN DIFFERENTIALLY ROTATING MAGNETIZED FLUIDS AT LOW MAGNETIC PRANDTL NUMBER

FARRUKH NAUMAN AND MARTIN E. PESSAH

Niels Bohr International Academy, The Niels Bohr Institute, Blegdamsvej 17, DK-2100, Copenhagen Ø, Denmark.

(Dated: September 28, 2016)

ABSTRACT

We show for the first time that sustained turbulence is possible at low magnetic Prandtl number for Keplerian flows with no mean magnetic flux. Our results indicate that increasing the vertical domain size is equivalent to increasing the dynamical range between the energy injection scale and the dissipative scale. This has important implications for a large variety of differentially rotating systems with low magnetic Prandtl number such as protostellar disks and laboratory experiments.

Keywords: accretion, accretion disks — magnetohydrodynamic turbulence — plasma, dynamo theory

1. INTRODUCTION

Differentially rotating flows are ubiquitous in nature, from accretion flows around young stars and compact objects to convection zones inside stars. Magnetic fields are important in many of these flows. Understanding the stability properties of these flows can shed light into the processes driving the transport of angular momentum and energy in these environments. The Magnetorotational Instability (MRI) is a linear instability that has emerged as a potential explanation for the origin of turbulence in weakly magnetized differentially rotating flows (e.g. magnetized Taylor-Couette flows, accretion disks [Velikhov \(1959\)](#); [Chandrasekhar \(1960\)](#); [Balbus & Hawley \(1991\)](#)). MRI requires a weak magnetic flux to work but given the paucity of direct observations of magnetic field amplitudes and geometries, it is of interest to explore a range of cases for the magnetic flux threading an accretion flow, including the case of zero mean magnetic flux.

The earliest 3D simulation of ideal magnetohydrodynamic (MHD) Keplerian shear flow without a mean magnetic flux by [Hawley et al. \(1996\)](#) showed that the saturated level of turbulent stresses decreases with resolution (see also [Pessah et al. \(2007\)](#)). Later work by [Fromang et al. \(2007\)](#) used physical diffusion coefficients and showed that turbulence does not persist for magnetic Prandtl number, $Pm = Rm/Re \lesssim 2$. Recent work on linearly stable hydrodynamic shear flows (e.g., [Hof et al. \(2006\)](#)) suggests that turbulence lifetime is finite and increases exponentially with the Reynolds number, Re . [Rempel et al. \(2010\)](#) explored the question of turbulence lifetime in Keplerian shear flows with zero net magnetic flux with a small vertical domain size ($L_z = 1$) and found that the turbulence lifetime increases exponentially with Rm between 9,000 and 11,000 ($Re = 3, 125$).

Examples of $Pm \ll 1$ systems include astrophysical systems such as protostellar disks (for more astrophysical examples, see table 1 of [Brandenburg & Subramanian \(2005\)](#)) and laboratory plasmas (e.g., [Sisan et al. \(2004\)](#), [Seilmayer et al. \(2014\)](#)). The $Pm \ll 1$ limit has been studied extensively in non-helically forced isotropic MHD turbulence simulations and theory, with the consensus that a larger Rm_{crit} is required in this limit compared to the $Pm \gg 1$ [Schekochihin et al. \(2007\)](#). Earlier work by [Schekochihin et al. \(2004\)](#) had claimed that the small scale dynamo (growth of the magnetic field on dissipation scales) does not exist in the low Pm limit but later work ([Iskakov et al. 2007](#)) demonstrated growth and sustenance of magnetic fields (at a higher Rm_{crit}). In their study, [Schekochihin et al. \(2004\)](#) conjectured that a low Pm dynamo might only be possible in the presence of a mean field. Indeed for MRI simulations with net magnetic flux, turbulence can be sustained in the low Pm regime ([Lesur & Longaretti \(2007\)](#), [Meheut et al. \(2015\)](#)).

The question of whether a large domain size can have a significant effect on flow stability has been long under discussion (e.g., [Pomeau \(1986\)](#), [Philip & Manneville \(2011\)](#)). The shearing box is a local approximation [Goldreich & Lynden-Bell \(1965\)](#) for differentially rotating flows such as accretion disks (or Taylor-Couette flows) and it is unclear whether it can exhibit similar behavior as the spatiotemporal chaotic patterns that might emerge in a realistic accretion flow with its very large spatial extent and very large Re ([Cross & Hohenberg \(1993\)](#)). Nevertheless, within the shearing box framework, it is of interest to explore how a small domain size (‘minimal flow unit’ [Jimenez & Moin \(1991\)](#), [Rincon et al. \(2007\)](#)) is different from a system with larger degrees of freedom as a result of larger domain size, resolution, Re and Rm . Using an asymptotic analysis with net toroidal and vertical flux, [Julien & Knobloch \(2007\)](#) showed that scale separation between the most unstable mode and the vertical domain size leads to a saturated state of MRI where the energy is dominated by the box scale in the vertical direction.

In this Letter, we explore the question of whether sustained turbulence can be found in unstratified¹ incompressible MHD Keplerian shear flows with zero mean magnetic flux with a variety of box sizes and dissipation coefficients. We find that the turbulence lifetime is very sensitive to the vertical box size² and that large scale structures develop both in the velocity and the magnetic fields.

2. NUMERICAL SIMULATIONS AND RESULTS

We use the publicly available pseudospectral code SNOOPY³ (Lesur & Longaretti (2007)). All of our simulations employ a zero net flux initial field $\mathbf{B}_{\text{ini}} = B_0 \sin(k_x x) \mathbf{e}_z$. The shear profile is $\mathbf{V}_{\text{sh}} = -Sx \mathbf{e}_y$, where $S = q\Omega = 1$ ($q = 1.5$ for Keplerian shear) is the shear parameter and Ω is the angular frequency. We apply perturbations of order LS to the first few velocity modes. All of our results are reported in units of shear times $1/S$ (for comparison, 10,000 shear times are equal to about 1,061 orbits ($2\pi/\Omega$) in our units). The dissipation coefficients are characterized by the Reynolds number, $Re = SL^2/\nu$ and the magnetic Reynolds number, $Rm = SL^2/\eta$; where $L = 1$. For most of our runs, $L_x = L$. The magnetic Prandtl number $Pm = Rm/Re = \nu/\eta$, is of course independent of the choice of characteristic scale. We report our domain sizes as $L_x \times L_y \times L_z$.

2.1. Fixed Re , Variable L_z and Pm

In the top panel of fig. 1, we find that as we increase L_z while keeping $L_x = 1$ and $L_y = 2$, even $Pm < 1$ manages to sustain turbulence for several thousand shear times. This is to be contrasted with earlier work of Fromang et al. (2007) ($L_z = 1, Re = 3, 125$) and Shi et al. (2016) ($L_z = 4, Re = 3, 125$), which showed that $Pm \geq 2$ is required for sustained turbulence. Shi et al. (2016) reports that $Pm = 2$ simulation sustained turbulence for thousands of shear times while their $Pm = 1$ run decayed after about 2,000 shear times. In an extensive numerical study, Rempel et al. (2010) (using SNOOPY) showed that turbulence is transient and the lifetime of turbulence increases exponentially with Rm (for $2.88 \leq Pm \leq 3.52$) with $L_z = 1, Re = 3, 125$. We do not conduct such an exhaustive study here to determine the functional dependence of lifetime on L_z and Pm , since a way of conducting such a study is to evolve a flow to a fully turbulent state and then take that state as initial condition for runs with different dissipation coefficients (Rempel et al. 2010). Due to the spatiotemporal chaotic nature of turbulence, it can take hundreds of runs for each L_z to get good statistics.

2.2. Fixed L_z , Variable Re and Pm

We show lifetime of the turbulent flow as a function of Re and Rm with fixed $L_z = 4$ in the bottom panel of fig.

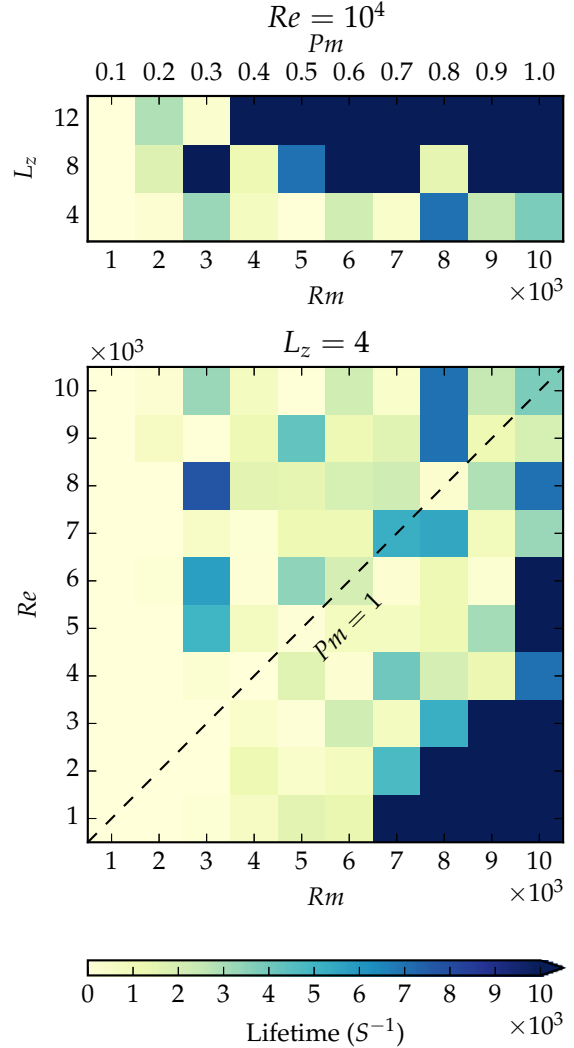


Figure 1. Turbulence lifetime in units of $1,000S^{-1}$ for runs with: (i) fixed $Re = 10,000$ but variable vertical box size L_z and $Pm = Rm/Re$ (top panel); (ii) fixed vertical box size $L_z = 4$ ($1 \times 2 \times 4$) but variable Re and Rm (bottom panel). The lifetime of turbulence increases with larger Re and Rm , but it is most sensitive to L_z . Note that we stopped all of our simulations at 10,000 shear times so the runs that show 10,000 shear times might have a much larger lifetime. The number of zones used for these simulations are $64 \times 64 \times (64 * L_z)$.

1. The lifetime increases with Pm (and Rm , the bottom right of the panel). More importantly, we find that at high enough Re , turbulence can last several thousand shear times at $Pm < 1$. Combined with fig. 1, this result hints that as $Re \rightarrow \infty$ and $L \gg 1$, Pm_{crit} reaches a low asymptotic value (Fromang et al. (2007)) and our work suggests that $Pm_{\text{crit}} < 1$ (Schekochihin et al. 2004).

Walker et al. (2016) did simulations with a domain size of $2 \times 4 \times 1$ at an unprecedented high resolution of $1024 \times 1024 \times 512$ where they fixed $Re = 45,000$. They found that zero net flux simulation immediately decays at $Pm = 1$. We used a lower resolution but larger vertical domain size L_z ranging from 1, 2, 4, 8 at fixed $Re = Rm = 10,000$. As shown in fig. 2, the $2 \times 4 \times 4$ run is turbulent for $2,000S^{-1}$ until it decays while the $2 \times 4 \times 8$ run remains turbulent for

¹ Davis et al. (2010) suggest that density stratification due to gravity lowers the critical Pm slightly but they did not explore the $Pm < 1$ regime.

² We do not explore elongated azimuthal domains as Riols et al. (2015) already explored $L_y = 20$ and did not find evidence for sustained turbulence at $Pm < 1$.

³ <http://ipag.osug.fr/~lesurg/snoopy.html>

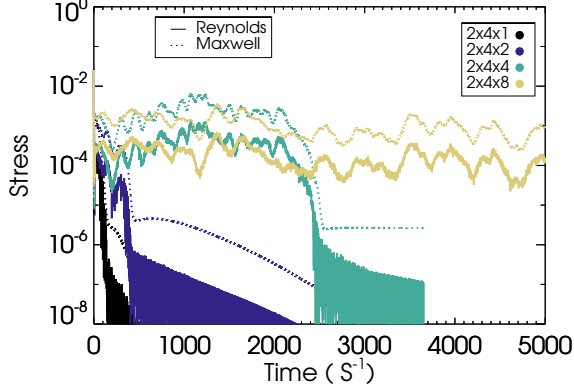


Figure 2. Time history for runs with the same horizontal domain size ($L_x = 2$, $L_y = 4$) as Walker et al. (2016) but larger vertical domains. We see sustained turbulence even though our $Re = Rm = 10,000$, much lower than that of Walker et al. (2016).

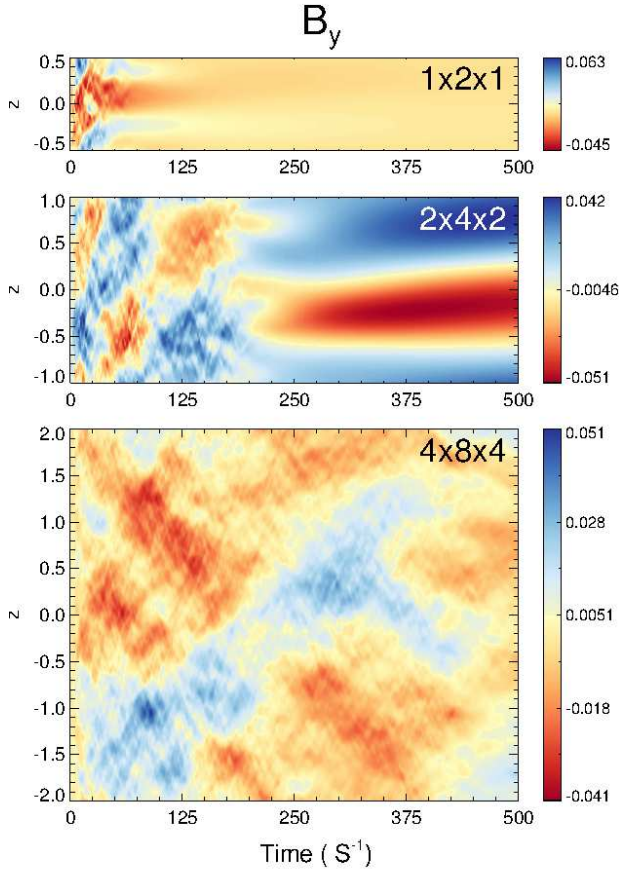


Figure 3. Comparison of the $\langle B_y \rangle$ (angled brackets represent xy average) profiles for three different box sizes $1 \times 2 \times 1$, $2 \times 4 \times 2$, $4 \times 8 \times 4$ with $Re = Rm = 10,000$ with the same aspect ratio. Turbulence is only sustained for significant duration for the largest domain, $4 \times 8 \times 4$.

the entire duration of our run (and possibly much longer), which is $10,000 S^{-1}$. This clearly demonstrates that vertical domain size is an important parameter in transition to turbulence studies in the shearing box.

2.3. Same aspect ratio ($L_z/L_x=1$), larger box size

We do not find any significant evidence for turbulence in a domain with size $1 \times 2 \times 1$ (with a resolution of 64^3) and short lived turbulence (~ 175 shear times) for $2 \times 4 \times 2$ (128^3). The $4 \times 8 \times 4$ (256^3) run is turbulent for nearly 5,000 thousand shear times. The dissipation coefficients are set by $Re = Rm = 10,000$, which is independent of the absolute box size since we scale both coefficients with a fixed $L = 1$. We plot the $\langle B_y \rangle$ (angled brackets represent xy average) for the three runs $1 \times 2 \times 1$, $2 \times 4 \times 2$ and $4 \times 8 \times 4$ in fig. 3. A coherent field develops in all cases but only the biggest box size manages to sustain turbulence for a long time. To compare the effects of increasing the domain size to the effect of increasing the resolution, we ran a simulation at $1 \times 2 \times 1$ with a resolution of 256^3 (the same number of zones used in the $4 \times 8 \times 4$) and found no signs of turbulence with $Re = Rm = 10,000$. This comparison suggests that the turbulence lifetime is more sensitive to the box size than the resolution.

2.4. Velocity fields

We plot the azimuthal velocity profile $\langle V_y \rangle$ in fig. 4 for three different vertical box sizes $L_z = 4, 8, 12$ while keeping $L_x = 1$, $L_y = 2$, $Re = Rm = 10,000$. The spatiotemporal dependence of the velocities suggest that velocities behave like: $\sin(\kappa t + \sin k_z z)$, where $k_z = 2\pi n_z/L_z$, and $2\pi/\kappa \sim 9.42 S^{-1}$ is the epicyclic time, with $\kappa^2 = 2\Omega^2(2 - q)$. The vertical profile of different velocity components seems to be dominated by $n_z = 1$ regardless of box size, which indicates that the gap between the energy containing scale and the dissipation scale increases with the increase in the vertical domain. The corresponding azimuthal magnetic field component profiles for these runs behave much in the same way as in fig. 3 so they are also dominated by $n_z = 1$ irrespective of the domain size. Our results are largely consistent with the recent suggestion by Sekimoto et al. (2016), that shearing box flow has the vertical box size as the outer scale.

2.5. Magnetic fields (dynamo)

Both Shi et al. (2016) and Walker et al. (2016) refer to ‘dynamo’ in their discussion, with the focus on the large scale in the former and small scale in the latter. One might ask which one of these two different mechanisms (or both) is responsible for the growth and sustenance of magnetic fields in a zero net flux MHD Keplerian flows. This distinction is difficult since it is not entirely clear whether there is a separation of *spatial* scales between the box scale and the forcing scale. An argument can be made that since the azimuthal magnetic field varies on several epicyclic times (e.g. $4 \times 8 \times 4$ run has a cycle period of about $500 S^{-1}$, see fig. 3) as opposed to one epicyclic time ($\sim 9.42 S^{-1}$) for the velocity field, there is at least a *temporal* scale separation⁴ and hence a large scale dynamo (Gressel & Pessah (2015), Bhat et al. (2016)).

The small scale dynamo, on the other hand, refers to the growth of magnetic fields on the viscous scale ($Pm > 1$)

⁴ We thank Eric Blackman for pointing this out.

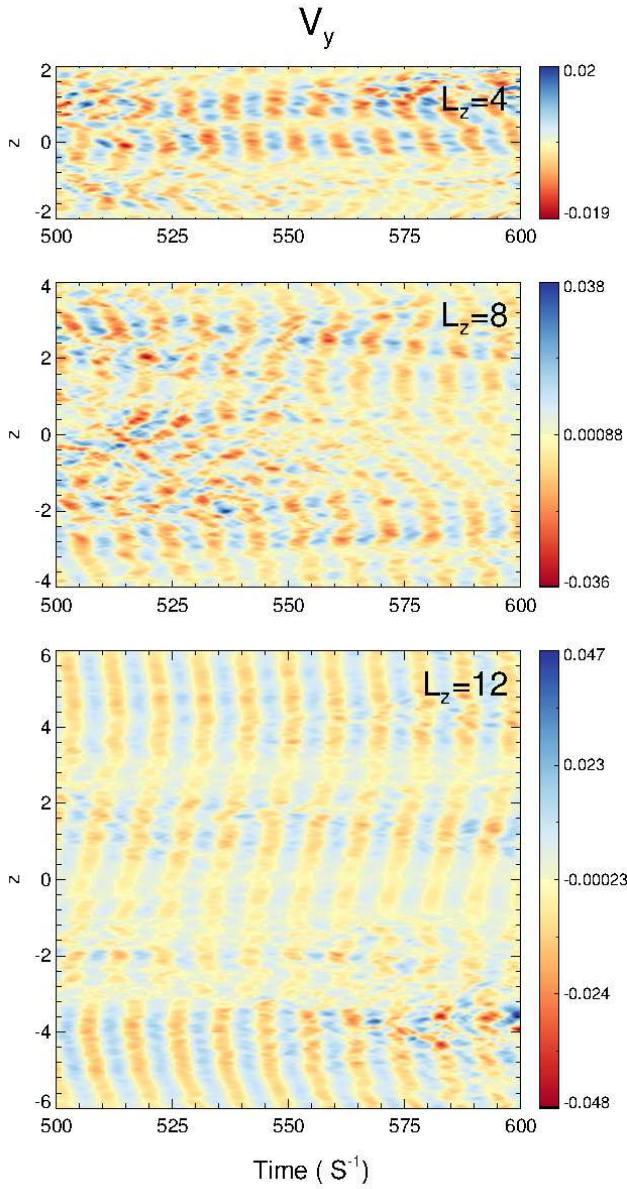


Figure 4. Comparison of the $\langle V_y \rangle$ profiles for three different vertical box sizes $1 \times 2 \times L_z$: $L_z = 4, 8, 12$ with $Re = Rm = 10,000$. A coherent velocity field ($\sim \sin(\kappa t + \sin k_z z)$), which is dominated by the vertical box scale suggests that the vertical box size has a very significant role in determining the (kinetic) ‘energy containing’ scale of the system.

or the resistive scale ($Pm < 1$) (Schekochihin et al. (2007)) and it might be the mechanism responsible for increasing the lifetime as Rm increases. There is yet another possibility: growth due to a mean velocity flow (e.g., Cattaneo & Tobias (2005), Mininni & Montgomery (2005)). This might be quite significant in the simulations we report here since the velocities seems to settle into large scale periodic structures (fig. 4). Schekochihin et al. (2007) suggest comparing the growth rate dependence of the magnetic field on Rm to distinguish between the small scale dynamo and the mean velocity flow

driven dynamo but since we trigger our simulations with finite amplitude perturbations, it is unclear how to determine the growth rates and hence distinguish between the two.

2.6. Resolution test

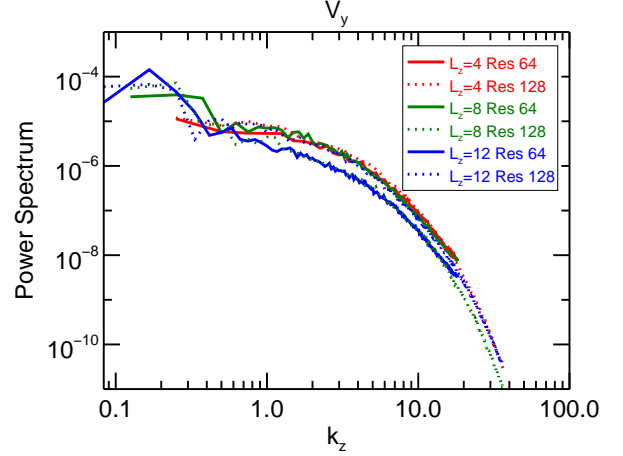


Figure 5. Power spectrum of $\langle V_y \rangle(k_z)$ averaged over $251 - 500 S^{-1}$, where $k_z = 2\pi n_z / L_z$ for $L_z = 4, 8, 12$ at two different resolutions $64 \times 64 \times (64 * L_z)$ and $128 \times 128 \times (128 * L_z)$. $Re = Rm = 10,000$ for all of these runs. This plot suggests that resolution does not significantly alter the azimuthal velocity as a function of resolution.

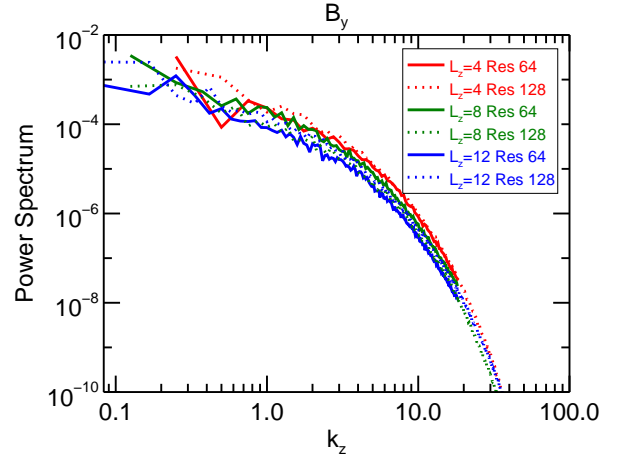


Figure 6. Same as fig. 5 but power spectrum of $\langle B_y \rangle(k_z)$.

As a resolution check, we compare the power spectrum in k_z ($= 2\pi n_z / L_z$) of three different runs $1 \times 2 \times L_z$ at $L_z = 4, 8, 12$ ($Re = Rm = 10,000$) at two different resolutions $64 \times 64 \times (64 * L_z)$ and $128 \times 128 \times (128 * L_z)$ in figs. 5 and 6. We find that the increase in resolution does not significantly alter the turbulent power spectrum. There is some discrepancy at low k_z but we point out that since both V_y and B_y show cyclic behavior, the chosen range and initial point for temporal averaging can have a significant effect on the largest wavelengths.

3. DISCUSSION AND CONCLUSIONS

Recent numerical simulations and experiments on hydrodynamic shear flows suggest that transition to turbulence mimics a second order phase transition and can be described by the directed percolation universality class (Shih et al. (2016); Lemoult et al. (2016); Sano & Tamai (2016)). The basic idea is that linearly stable flows develop two kinds of domains: laminar and turbulent corresponding to dead and active states. It is through the interaction of these two types of domains that turbulence starts to spread and eventually fills the whole domain as the Reynolds number is increased. This might explain why the box size plays a special role: a larger box size allows for more turbulent and laminar domains to fit in the box and thus allows for complex pattern formation (Cross & Hohenberg (1993)). A potential analogy for our simulations is the generation of B_x (active state) through stochastic processes on small scales and the subsequent generation of B_y (dead state) through the SB_x term (see also the recent study by Riols et al. (2016) who describe the dynamo using the concept of ‘active’ and ‘slave’ perturbations).

We have demonstrated for the first time that Keplerian shear MHD turbulence (without using any external forcing or net magnetic flux) can sustain for several thousand shear

times at $Pm \leq 1$ (for $Re = 10,000$), if one uses domain sizes with $L_z \geq 4$. We find that with the Re and Rm up to 10,000 there is no turbulence at $Pm \leq 1$ when $L_z = 1$, consistent with previous work. It seems that by increasing L_z , one increases the separation between the outer scale ($\sim L_z$) and the dissipation scale, which is qualitatively similar to the effect of increasing Rm . But the increase in L_z at fixed Re and Rm has a more dramatic effect on turbulence lifetime than increasing Re and Rm at $L_z = 1$. Our work clearly illustrates that flow stability studies should not be confined to very small domains and has especially important implications for laboratory plasmas, which can only explore the $Pm \ll 1$ regime Sisan et al. (2004), Seilmayer et al. (2014).

We thank Shantanu Agarwal, Eric Blackman, Oliver Gressel, Tobias Heinemann, Paul Manneville, Jiming Shi, Jim Stone and Francois Rincon for discussions. Colin McNally is thanked for suggestions and help with the figures. The computations were performed on the BlueStreak cluster at the Center for Integrated Research Computing (CIRC) at the University of Rochester. The research leading to these results has received funding from the European Research Council under the European Unions Seventh Framework Programme (FP/2007-2013) under ERC grant agreement 306614.

REFERENCES

- Balbus, S. A., & Hawley, J. F. 1991, *ApJ*, 376, 214
 Bhat, P., Ebrahimi, F., & Blackman, E. G. 2016, *MNRAS*, 462, 818
 Brandenburg, A., & Subramanian, K. 2005, *PhR*, 417, 1
 Cattaneo, F., & Tobias, S. M. 2005, *Physics of Fluids*, 17, 127105
 Chandrasekhar, S. 1960, *Proceedings of the National Academy of Science*, 46, 253
 Cross, M. C., & Hohenberg, P. C. 1993, *Rev. Mod. Phys.*, 65, 851
 Davis, S. W., Stone, J. M., & Pessah, M. E. 2010, *ApJ*, 713, 52
 Fromang, S., Papaloizou, J., Lesur, G., & Heinemann, T. 2007, *A&A*, 476, 1123
 Goldreich, P., & Lynden-Bell, D. 1965, *MNRAS*, 130, 97
 Gressel, O., & Pessah, M. E. 2015, *ApJ*, 810, 59
 Hawley, J. F., Gammie, C. F., & Balbus, S. A. 1996, *ApJ*, 464, 690
 Hof, B., Westerweel, J., Schneider, T. M., & Eckhardt, B. 2006, *Nature*, 443, 59
 Isakov, A. B., Schekochihin, A. A., Cowley, S. C., McWilliams, J. C., & Proctor, M. R. E. 2007, *Physical Review Letters*, 98, 208501
 Jimenez, J., & Moin, P. 1991, *Journal of Fluid Mechanics*, 225, 213
 Julien, K., & Knobloch, E. 2007, *JMP*, 48, 065405
 Lemoult, G., Shi, L., Avila, K., et al. 2016, *Nature Physics*, 12, 254
 Lesur, G., & Longaretti, P.-Y. 2007, *MNRAS*, 378, 1471
 Meheut, H., Fromang, S., Lesur, G., Joos, M., & Longaretti, P.-Y. 2015, *A&A*, 579, A117
 Mininni, P. D., & Montgomery, D. C. 2005, *PhRvE*, 72, 056320
 Pessah, M. E., Chan, C.-k., & Psaltis, D. 2007, *ApJL*, 668, L51
 Philip, J., & Manneville, P. 2011, *PhRvE*, 83, 036308
 Pomeau, Y. 1986, *Physica D: Nonlinear Phenomena*, 23, 3
 Rempel, E. L., Lesur, G., & Proctor, M. R. E. 2010, *Physical Review Letters*, 105, 044501
 Rincon, F., Ogilvie, G. I., & Cossu, C. 2007, *A&A*, 463, 817
 Riols, A., Rincon, F., Cossu, C., et al. 2016, *ArXiv e-prints*, arXiv:1607.02903
 —. 2015, *A&A*, 575, A14
 Sano, M., & Tamai, K. 2016, *Nature Physics*, 12, 249
 Schekochihin, A. A., Cowley, S. C., Maron, J. L., & McWilliams, J. C. 2004, *Physical Review Letters*, 92, 054502
 Schekochihin, A. A., Isakov, A. B., Cowley, S. C., et al. 2007, *New Journal of Physics*, 9, 300
 Seilmayer, M., Galindo, V., Gerbeth, G., et al. 2014, *Physical Review Letters*, 113, 024505
 Sekimoto, A., Dong, S., & Jiménez, J. 2016, *Physics of Fluids*, 28, 035101
 Shi, J.-M., Stone, J. M., & Huang, C. X. 2016, *MNRAS*, 456, 2273
 Shih, H.-Y., Hsieh, T.-L., & Goldenfeld, N. 2016, *Nature Physics*, 12, 245
 Sisan, D. R., Mujica, N., Tillotson, W. A., et al. 2004, *Physical Review Letters*, 93, 114502
 Velikhov, E. P. 1959, *JETP*, 36, 995
 Walker, J., Lesur, G., & Boldyrev, S. 2016, *MNRAS*, 457, L39